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Doubled Resonances in the Eigenchannel Representation*

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The eigenamplitudes of the unitary doubled-resonance amplitude for the two-channel case are computed and discussed. In addition to poles, the eigenamplitudes also have square-root branch points in the complex energy plane.

THE constraints of unitarity on a partial-wave amplitude with two resonance poles have been solved in the many two-body channel case.[1–4] The calculation was performed in channel space. In outline, we assumed that the residues of the poles factored when the poles did not coincide. In the limit in which the two poles approach each other, the residues blow up in a well-defined way. In general, in this limit the amplitude becomes a pole with a nonfactorizable residue, plus a double-pole term with a factorizable coefficient. The result is the most general parametrization of a "dipole" which is consistent with two-body unitarity.

At first sight, it might appear that working in channel space is uneconomical, since one might expect the eigenamplitudes to be particularly simple in form. For example, one might guess that two of the eigenamplitudes are simple poles, or that one eigenamplitude has a double pole, and that the rotation back to channel space is energy-independent. However, there is a serious difficulty with such models: There is no reason to expect the eigenamplitudes to have the analytic properties usually ascribed to the amplitude in channel space. There can be singularities in the eigenamplitudes that are canceled by singularities in the matrix that rotates the eigenamplitudes to channel space. It turns out that the eigenamplitudes for the "interesting" parametrizations of the doubled resonance do have singularities that are not present in the amplitude in channel space.[5] Of course, in these cases the rotation is energy-dependent, and has corresponding singularities. The object of this paper is to compute the eigenamplitudes in the two-channel case, and to show that the "extra" singularities must be present in order to preserve the factorization properties of the double-resonance amplitude. As has been noted for the case of a simple resonance plus background, the usefulness of the eigenphase representation is greatly impaired by the complicated nature of the eigenamplitudes.[5]

We consider a unitary two-channel amplitude in which two resonance poles coincide. The partial-wave amplitude can be written[1]

$$A_{ij}=-\frac{\Gamma(X_iX_j+U_iU_j)}{E-M+i\Gamma}-\xi\Gamma^2e^{-i\theta}\frac{(X_i+iU_i)(X_j+iU_j)}{(E-M+i\Gamma)^2}, \quad (1)$$

where the amplitude A_{ij} is related to the S matrix by

$$S_{ij}=\delta_{ij}+2iA_{ij}. \quad (2)$$

The width Γ is half the usual width, θ is the angle of the dipole in the complex energy plane measured with respect to the real axis, and ξ is a "dipole strength parameter," which arises in the taking the limit of the coincidence of the two resonance poles. It is constrained by $0\leq\xi\leq 1$. The couplings X_i and U_i are real, and by unitarity they satisfy the conditions

$$X_1^2+X_2^2=1+\xi\sin\theta, \quad (3a)$$

$$U_1^2+U_2^2=1-\xi\sin\theta, \quad (3b)$$

$$X_1U_1+X_2U_2=-\xi\cos\theta. \quad (3c)$$

It is straightforward to compute the eigenvalues $S_\pm$ from Eqs. (1)–(3). The eigenvalues depend only on E, M, ξ, and Γ:

$$S_\pm=e^{2i\delta_\pm}=\left[\frac{[(E-M)^2+(1-\xi^2)\Gamma^2]^{1/2}\pm i\xi\Gamma}{E-M+i\Gamma}\right]^2. \quad (4)$$

The eigenamplitudes are easily obtained from Eq. (4). It is somewhat easier to discuss the results in terms of

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[5] The existence of "extra" singularities in the eigenamplitudes in the case of a simple resonance interfering with an energy-independent background has been discussed by H. A. Weidenmueller, Phys. Letters **24B**, 441 (1967); C. J. Goebel and K. W. McVoy, Phys. Rev. **164**, 1932 (1967).

the K-matrix eigenamplitudes,

$$\tan\delta_{\pm} = \frac{\Gamma\{-(E-M)\pm\xi[(E-M)^2+(1-\xi^2)\Gamma^2]^{1/2}\}}{(E-M+\xi\Gamma)(E-M-\xi\Gamma)}. \quad (5)$$

Equation (5) has the following singularity structure: $\tan\delta_+$ has a pole at $E=M-\xi\Gamma$ (but no pole at $E=M+\xi\Gamma$); $\tan\delta_-$ has a pole at $E=M+\xi\Gamma$; and $\tan\delta_{\pm}$ both have square-root branch points in the complex energy plane at $E=M\pm i[(1-\xi^2)\Gamma^2]^{1/2}$. The branch points may come as a surprise, but there is no reason *a priori* to expect the eigenamplitudes to have the same analytic structure as the channel amplitudes.

We now examine Eq. (5) in the limits, $\xi\rightarrow 0$ and $\xi\rightarrow 1$. In the limit $\xi\rightarrow 0$, both eigenamplitudes are simple poles,

$$\tan\delta_{\pm} = -\Gamma/(E-M). \quad (6)$$

From Eq. (3) it is clear that this is just the case of two resonances in orthogonal channels, and the resonances are in no way coupled. Of course, the partial-wave cross sections only have single peaks. This case corresponds physically to two resonances with different quantum numbers.

In the limit $\xi\rightarrow 1$, Eq. (5) becomes

$$\tan\delta_+ = \frac{-2\Gamma(E-M)}{(E-M)^2-\Gamma^2}, \quad E<M$$

$$\tan\delta_- = 0,$$

and

$$\tan\delta_+ = 0,$$

$$\tan\delta_- = \frac{-2\Gamma(E-M)}{(E-M)^2-\Gamma^2}, \quad E>M. \quad (7)$$

Each of the eigenphases rises through $\frac{1}{2}\pi$, but $\tan\delta_{\pm}$ is not analytic in the complex energy plane. When Eq. (7) is rotated back to channel space, the amplitude has the form

$$A_{ij} = \left[-\frac{2\Gamma}{E-M+i\Gamma} + \frac{2i\Gamma^2}{(E-M+i\Gamma)^2}\right]\frac{X_iX_j}{1+\sin\theta}. \quad (8)$$

Physically, this is the case where doubled peaks occur in all partial-wave cross sections, and the amplitude is a single-channel doubled resonance multiplied by a factorizable matrix. Thus the limits $\xi\rightarrow 0$ and $\xi\rightarrow 1$ have simple eigenamplitude representations, but if $0<\xi<1$, then the eigenamplitudes are not simple, since they possess singularities not present in the channel amplitudes. Without a knowledge of the analytic structure of the eigenamplitudes, it is difficult to find the most general parametrization of a unitary amplitude.

We conclude by presenting the matrix that transforms the amplitude into the eigenamplitude representation. The rotation is defined by

$$S_{\mathrm{diag}} = U^{\dagger}SU, \quad (9)$$

where U is a 2×2 orthogonal matrix,

$$U = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \quad (10)$$

The rotation angle is

$$\sin 2\alpha = d/\sigma,$$

where

$$\sigma = 2\xi[(E-M)^2+(1-\xi^2)\Gamma^2]^{1/2},$$

$$d = 2(E-M)(X_1X_2+U_1U_2)+\Gamma(X_1U_2-X_2U_1) \times (X_1^2-X_2^2+U_1^2-U_2^2). \quad (11)$$

It is interesting to examine the rotation in the limit $\xi\rightarrow 1$. When $\xi=1$, $X_1U_2-X_2U_1=0$; Eq. (11) then gives

$$\sin 2\alpha = \frac{\mathrm{sgn}(E-M)}{X_1X_2+U_1U_2}. \quad (12)$$

The rotation U is nonanalytic at $E=M$, which was expected, since the amplitude is analytic at the same point, but the eigenamplitudes are not.

The discontinuity in U at $E=M$ is equivalent to a relabeling of the eigenamplitudes. Then one of the phase shifts increases continuously from 0 to 2π, and the other is constantly zero, which is what one would obtain from a direct analysis of the factorized expression given in Eq. (8).

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